Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00070-9

J. Appl. Maths Mechs, Vol. 66, No. 4, pp. 523–527, 2002

0021-8928/02/$—see front matter

A MODEL OF THE MECHANISM OF MOTION OF A SNAKE†

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(*Received* 3 *April* 2001)

The motion of a snake is considered using as a model an elastic rod in which the shape of the neutral axis is controllable. The optimal control of this shape, guaranteeing the minimum internal effort for a given value of the average tractive force, is found.

There are different ideas of the mechanism that enables snakes to move. One of these is the so-called "wave-like mass transfer" [1], when the snake shifts different parts of its body in the manner of a "relay race", and the friction forces at the instantaneous points of support are such as to guarantee that there is no local slipping. A recent paper [2] has assigned dry friction a more significant role: phases of rest alternate with phases of sliding, yielding an intermittent motion in which inertial effects play an important role.

The principle that the tractive force in the motion of a snake is created entirely by internal forces, without the participation of friction, was first formulated by M. A. Lavrent'yev (see, e.g. [3]).

In the model proposed below, dry friction also plays a passive role: it exists, if at all, only as resistance to the motion, which must be overcome. The tractive force is produced by the snake by adjusting the shape of its body as the latter interacts with the surrounding obstacles. This model resembles those of [3–6], in which the snake is regarded as an absolutely flexible inextensible thread with a control torque distributed over its length, the snake being treated either from the start as a continuous medium [3–5], or first as a multilink structure, from which a continuous-medium model is then obtained by taking the limit [6].

Observations of nature indicate that snakes move in such a way that each point of the axial line of the snake's body describes the same trajectory as all other such points. In other words, the velocity vector of the snake at any point is directed along the tangent to the trajectory or to the curved line of the body. Such motion of a flexible thread is known as apparent rest.

This means that the trajectory of a snake, the axial line of whose body is at each instant of time a whole piece of the trajectory, may be considered as a geometrical constraint. Moving according to the constraint, the snake is arranged by internal forces so that the reaction of the constraint has the required direction.

A simple example will explain this idea (Fig. 1). Suppose three point masses A, B and C are situated all the time on a fixed parabola (the constraint). They are linked together by rigid weightless rods with an elastic element (hinge) at the points of articulation. The points may move without friction along the parabola. There are no other forces.

If the rods are of the same length, there is an obvious equilibrium position in which the point B lies at the origin. If the hinges of the rods in the stress-free state are collinear, the potential energy of the deformed state is a maximum in the equilibrium position just described and the equilibrium is unstable. The system, if disturbed, will go off to infinity. If the hinge is stress-free at this very equilibrium position, the latter will obviously be stable. The hinge may be stress-free at any position of the point B on the parabola. Then the system will have two stable equilibrium positions, symmetric to one another relative to the y axis, and one unstable equilibrium position at the origin.

Suppose μ is the angle between the rods, as shown in Fig. 1, μ_0 is the value of μ at which the hinge is stress-free, and μ^* is the angle between the rods when the point B is the origin. The curves in Fig. 2 represent the behaviour of the potential energy as a function of the coordinate of the point B on the x axis, for different values of μ_0.

Thus, by varying the point at which the hinge on one of the links is attached, one can control both the equilibrium position and the motion of the system.

†*Prikl. Mat. Mekh.* Vol. 66, No. 4, pp. 534–538, 2002.

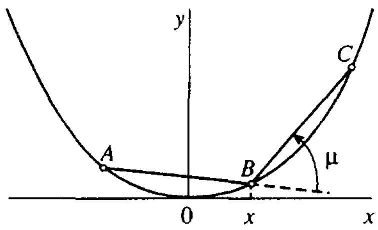

Fig. 1

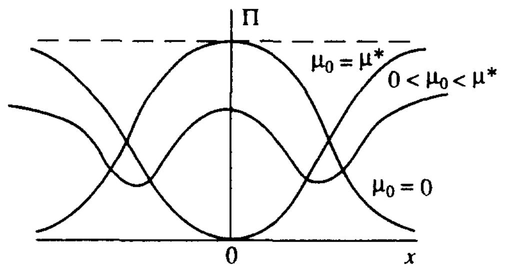

Fig. 2

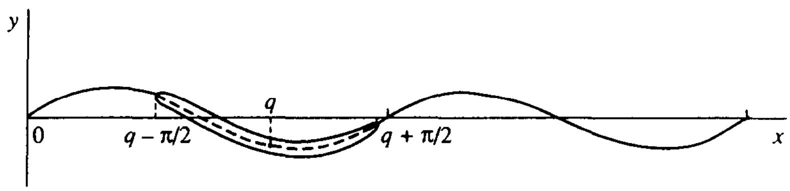

Fig. 3

The example may be complicated slightly by placing several articulated rods on the parabola with elastic elements between them. In such a system there will be several control parameters: the control may be applied to the stress-free relative position of each pair of rods. The number of control parameters may be diminished – in particular, reduced to one – by coupling together controls of different parameters.

The system under consideration is the simplest example of an elastic system in which, by internally regulating the parameters of an elastic element, one can control the motion of the system subject to an elastic constraint.

This is the basic idea of our model for the dynamic behaviour of a snake, which differs from the simple example just considered only in that it is a distributed elastic system. If the aim of the motion is to move from an initial point on a horizontal plane to a final point, the mechanism in question cannot perform the motion when the geometric constraint is chosen to be a straight line connecting those two points. Besides, on the basis of observations of nature, we shall assume from the start that the geometric constraint formed to achieve the aforementioned aim is a periodic curve, which may be considered, in the simplest case, as a sine curve (Fig. 3). The period and amplitude of the sine curve remain to be determined.

We may assumed without loss of generality that the length of the snake is π. The snake is a system with one degree of freedom, assumed moreover to be inextensible; a single generalized coordinate q_1 defines the position of the centre of the snake's body on the x axis. Its transversal measurements will be ignored.

We can simplify the analysis still further by assuming that the curvature of the trajectory is not large, so that it may be approximated by the second derivative: $y'' = d^2y/dx^2$. Then the potential energy of the deformed state of the snake may be computed as

$$\Pi = \frac{1}{2}EJ\int\limits_{q_1-\pi/2}^{q_1+\pi/2}\xi^2(x,x^*,q_2)dx,\quad \xi(x,x^*,q_2) = y''(x) - u''(x^*,q_2)$$

where E is the modulus of elasticity, J is the moment of inertia of the cross-section, and $u(x^*, q_2)$ is a function defining the shape of the snake's body in the stress-free state. The variable $x^* = x - q_1$ is the relative coordinate of a point on the body, and q_2 is a control parameter. The role of the control function in the distributed system, $u(x^*, q_2)$, is equivalent to that of the parameter μ_0 in the simple example considered previously.

The derivative of the force function $U(q_1, q_2) = -\Pi(q_1, q_2)$ with respect to the parameter q_1, which defines the position of the centre of the snake's body along the x axis, with the control function $u(x^*, q_2)$ frozen, is equal to the force exerted on the snake by the constraint

$$Q_1 = \frac{\partial U}{\partial q_1} = \frac{1}{2}EJ\left[\xi^2\left(q_1 - \frac{\pi}{2}, -\frac{\pi}{2}, q_2\right) - \xi^2\left(q_1 + \frac{\pi}{2}, \frac{\pi}{2}, q_2\right)\right] -$$
$$-EJ\int\limits_{q_1-\pi/2}^{q_1+\pi/2} \xi(x, x - q_1, q_2)u'''(x - q_1, q_2)dx$$

The derivative of the force function with respect to the parameter q_2 with q_1 fixed is a measure of the force exerted by the snake in implementing the control function. It is assumed that the electrical signals directed toward the snake's muscles are interconnected in such a way that the control of the neutral axis is achieved by regulation of a single scalar parameter (the snake does not control each muscle separately). Then a measure of the internal forces developed by the snake during the motion will be the increase in the energy of its deformed state when the control parameter is varied.

Before computing this force, it is convenient to write the integrand in the expression for the force function in terms of the relative coordinate; after that we find that

$$Q_2 = \frac{\partial U}{\partial q_2} = EJ\int\limits_{-\pi/2}^{\pi/2} \xi(x^* + q_1, x^*, q_2)\frac{\partial u''}{\partial q_2}dx^*$$

We formulate the following problem. It is required to find a control $u(x^*, q_2)$ of the neutral axis of the snake, moving along a sine curve, which, for a given average tractive force $\langle Q_1\rangle$, minimizes the control forces exerted, in the sense that $\langle Q_2^2\rangle \to \min$ (angular brackets denote the average with respect to q_1).

As already remarked, the snake is moving along a sine curve $y = a\sin bx$ whose amplitude a and spatial frequency b are not known in advance.

We shall also seek the control function as a sine curve, characterizing the propagation of the elastic wave along the snake's body:

$$u = c\sin\eta(x^*, q_2), \quad \eta(x^*, q_2) = dx^* + eq_2 + f$$

This function defines the propagation of an elastic wave tracing the motion of the snake's body along the constraint, if the control parameter q_2 follows the parameter of the motion $q_1 : q_2 = q_1 = q$.

Thus, we are looking for six parameters: a, b, c, d, e and f. We may assume without loss of generality that $a > 0, b > 0, c > 0$.

Let us evaluate the tractive force:

$$Q_1 = \frac{1}{2}EJ\left[\xi^2\left(q - \frac{\pi}{2}, -\frac{\pi}{2}, q\right) - \xi^2\left(q + \frac{\pi}{2}, \frac{\pi}{2}, q\right)\right] -$$
$$-EJ\int\limits_{q-\pi/2}^{q+\pi/2} \xi(x, x - q, q)cd^3\cos\eta(x - q, q)dx$$

$$\xi(x, x - q, q) = ab^2\sin bx - cd^2\sin\eta(x - q, q)$$

If there is no control ($c = 0$, the shape of the snake in the undeformed state is a straight line), the force acting on the snake has the form

$$Q_1 = -\frac{1}{2}a^2b^4EJ\sin 2bq\sin b\pi$$

For example, if the length of the snake is a quarter of a period of the trajectory, that is, $b = 1/2$, then $Q_1 = -(a^2EJ/32)\sin q$. This means that in the case of a sinusoidal constraint with no control, the snake is a mathematical pendulum, in which the role of the angular variable is played by the coordinate q, with alternating stable and unstable equilibrium positions. The same is true when the snake's length exceeds a quarter of a period by a multiple of a half-period. But if the snake's length is a multiple of the half-period of the trajectory, the force Q_1 will vanish identically as a function of q. In that case the snake will experience no forces due to the constraint, and any position on the constraint will be a neutral equilibrium position.

This last condition must be satisfied by the snake's length (or, equivalently, if the length is given, it must be satisfied by the trajectory chosen) so that the forces developed when a control is applied will not be consumed in overcoming a conservative force.

We have thus established that from now on, that is, when $c \neq 0$, the only meaningful case to be considered is when $b \in N$ ($b = 1, 2, \ldots$).

Computation of the average value of Q_1 leads in the general case to the expression

$$\langle Q_1\rangle = \begin{cases}\langle Q_1\rangle_+, & e+b=0,\\ \langle Q_1\rangle_-, & e-b=0,\end{cases} \quad \langle Q_1\rangle_\pm = EJab^3cd^2\omega_\pm \sin f$$

$$\omega_\pm = \frac{1}{b\pm d}\sin(b\pm d)\frac{\pi}{2}$$

Comparing the displacement wave $y = a\sin b(x^* + q)$ with the deformation wave of the neutral axis $u = b\sin\eta(x^*, q)$, we note that these waves move in opposite directions if $\operatorname{sgn}(ed) < 0$ and in the same direction if $\operatorname{sgn}(ed) > 0$. Therefore, if we consider $d > 0$ (this will not affect the generality of the argument), the expression obtained for $\langle Q_1\rangle$ may be expressed in the equivalent form

$$\langle Q_1\rangle = \begin{cases}\langle Q_1\rangle_+, & e<0\\ \langle Q_1\rangle_-, & e>0\end{cases}$$

It is obvious that, other conditions being equal, the tractive force is larger if both waves propagate in the same direction ($e > 0$). This will be the case considered below.

We now calculate $\langle Q_2^2\rangle$

$$\langle Q_2^2\rangle = \frac{1}{2}(EJab^3cd^2)^2(\omega_+^2 + 2\omega_-^2\sin^2 f) + \frac{1}{8}(EJbcd^3)^2\sin^2 d\pi -$$
$$-\frac{1}{2}E^2J^2ab^4c^3d^2\omega_+\sin d\pi\cos f$$

In what follows, to avoid cumbersome derivations, we will adopt a certain obvious assumption without formal justification. We shall assume that d is an integer. Then the expression for $\langle Q_2^2\rangle$ can be simplified; taking the expression for $\langle Q_1\rangle$ into account, we obtain

$$\langle Q_2^2\rangle = \langle Q_1\rangle^2\left[1 + \left(\frac{\omega_+}{\omega_-\sin f}\right)^2\right]$$

Here it is already easy to see that, for a given average tractive force $\langle Q_1\rangle$, the forces exerted $\langle Q_2^2\rangle$ will be minimal at $b = d$, when $\langle Q_2^2\rangle = \langle Q_1\rangle^2$. The expression for the average force under those conditions will be $\langle Q_1\rangle = (\pi/2)EJab^5c\sin f$. Obviously, it is advantageous to take the phase displacement f to be equal to $\pi/2$.

We have thus established that, in motion along the constraint $y = a\sin bx$, the optimal control of the neutral line in the snake's body has the form $u = c\cos b(x^* + q)$.

In conclusion, we will present the expression obtained for the treative force in dimensional variables, assuming that the dimensionless coordinate x is related to the corresponding dimensional coordinate $\tilde{x}$ by the formula $x = \pi\tilde{x}/L$, where L is the length of the snake. We have $\langle Q_1\rangle = \pi^5EJacb^5/(2L^4)$.

Example. Suppose the length of the snake is $L = 100$ cm. The average radius of the cross-section is $R = 2$ cm. The amplitude of the trajectory is $a = 5$ cm. The amplitude of the controlled neutral axis is $c = 1$ cm. Set

$b = 5$ (2.5 waves along the length of the body). The modulus of elasticity $E = 2$ kg/cm^2. For these data, the weight of the snake is 1.5 kg, and $J = 64/3$ cm^4. Substituting these figures into the last formula, we obtain $\langle Q_1 \rangle \cong 1$ kg.

The variation of the coordinate q with time depends on the formulation of the dynamical problem. In particular, if the tractive force cancels out the friction force, the coordinate is a linear function of time: $q = vt$. The change in the shape of the snake in dimensional variables and in an attached system of coordinates, will then be described by the formula $y = a\sin[\pi b(x^* + vt)/L]$. The frequency of the oscillations of the body is $\pi bv/L$. For the data of the example, taking the oscillation frequency to be equal to 2.5 Hz, we find the snake's speed to be 1 m/sec.

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Translated by D.L.